

HÖLDER REGULARITY FOR NON DIVERGENCE FORM ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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ABSTRACT. In this note we study the global regularity in the Morrey spaces $L^{p,\lambda}$ for the second derivatives for the strong solutions of non variational elliptic equations.

1. INTRODUCTION

The aim of this note is to study the global Morrey regularity for the second derivatives of the strong solutions of non variational elliptic equations. Namely, given a bounded domain $\Omega \subset \mathbb{R}^n$, we consider the linear equation

$$a_{ij}u_{x_i x_j} + b_i u_{x_i} + cu = f,$$

where the lower order coefficients b, c and f are assumed in the Morrey space $L^{p,\lambda}$ ($1 < p < +\infty, n-p < \lambda < n$) and the coefficients of the leading part are assumed in the class $VMO \cap L^\infty$.

As a consequence of our $W^{2,p}$ estimate (see Section 3) we obtain the Hölder continuity of the gradient.

The technique we use is quite simple; it is based on a multiplicative inequality for functions in Morrey classes combined with an iterative procedure.

The same problem has been studied in several papers by many Authors. Among them, we cite [1] and [2] where, in the case $c = b = 0$, Caffarelli proved that if f belongs to the Morrey space $L^{n,n\alpha}$, with $0 < \alpha < 1$, then every $W^{2,p}$ -viscosity solution u is of class $C^{1,\alpha}$. Subsequently, in [6] and [7], Caffarelli result were improved in a special case obtaining gradient regularity for any p . The result in [6] was obtained via a representation formula for the second derivatives of the solutions used in [4] and the study of some non convolution type integral operators. The result in [7] can be recovered from our by letting b and c identically zero.

2. PRELIMINARIES

Let Ω be a bounded open set in \mathbb{R}^n ($n \geq 3$). If $f \in L^1(\Omega)$ and $E \subset \Omega$ we set $f_E = \frac{1}{|E|} \int_E f dx$.

We recall some classical definitions.

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Definition 2.1. Let $1 \leq p < +\infty$, $0 < \lambda < n$, Ω a bounded domain in \mathbb{R}^n . A function $f \in L^p(\Omega)$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$ if

$$\|f\|_{L^{p,\lambda}(\Omega)}^p = \sup r^{-\lambda} \int_{B_r(x_0) \cap \Omega} |f|^p dx < +\infty,$$

the supremum being taken over $x_0 \in \Omega$ and $0 < r \leq \text{diam}\Omega$.

It is well known that $L^{p,\lambda}(\Omega)$ is a Banach space endowed with the above norm.

Definition 2.2. Let $1 \leq p < +\infty$, $0 < \lambda < n + p$, Ω a bounded domain in \mathbb{R}^n . A function $f \in L^p(\Omega)$ belongs to the Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$ if

$$(1) \quad [f]_{\mathcal{L}^{p,\lambda}(\Omega)}^p = \sup r^{-\lambda} \int_{B_r(x_0) \cap \Omega} |f - f_{B_r(x_0) \cap \Omega}|^p dx < +\infty,$$

the supremum being taken over $x_0 \in \Omega$ and $0 < r \leq \text{diam}\Omega$.

If $\lambda = n$ the definition gives back the definition of the space BMO .

Definition 2.3. Let Ω be a bounded domain in \mathbb{R}^n . A function $f \in BMO(\Omega)$ belongs to $VMO(\Omega)$ if

$$\eta(r) = \sup \int_{B_\rho(x_0) \cap \Omega} |f - f_{B_\rho(x_0) \cap \Omega}| dx$$

vanishes as $r \rightarrow 0^+$. Here the supremum is taken over $x_0 \in \Omega$ and $0 < \rho < r$.

The space $\mathcal{L}^{p,\lambda}$ is a Banach space endowed with the norm

$$\|f\|_{\mathcal{L}^{p,\lambda}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{\mathcal{L}^{p,\lambda}(\Omega)},$$

where $[f]_{\mathcal{L}^{p,\lambda}(\Omega)}$ is given by (1).

Definition 2.4. Let Ω be a bounded domain in \mathbb{R}^n . We say that Ω satisfies the condition K if there exists a positive constant $K > 0$ such that

$$|B_r(x_0) \cap \Omega| \geq Kr^n,$$

$\forall x_0 \in \Omega$ and $0 < r \leq \text{diam}\Omega$.

Remark 2.1. Any Lipschitz domain satisfies the condition K .

We state some useful results we need in the sequel.

Theorem 2.1. ([3], Theorem 2.1). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying condition K . Then

1. If $0 < \lambda < n$, $\mathcal{L}^{p,\lambda}(\Omega) = L^{p,\lambda}(\Omega)$ and there exist two positive constants C_1 and C_2 such that

$$C_1[f]_{\mathcal{L}^{p,\lambda}} \leq \|f\|_{L^{p,\lambda}} \leq C_2\|f\|_{\mathcal{L}^{p,\lambda}}.$$

2. If $n < \lambda \leq n + p$, $\mathcal{L}^{p,\lambda}(\Omega) = C^{0,\gamma}(\overline{\Omega})$, with $\gamma = \frac{\lambda-n}{p}$, and there exist two positive constants C_3 and C_4 such that

$$C_3[f]_{\mathcal{L}^{p,\lambda}} \leq [f]_{\gamma} \leq C_4[f]_{\mathcal{L}^{p,\lambda}},$$

where $[f]_{\gamma} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\gamma}}.$

In the sequel we will use the following multiplicative inequality.

Lemma 2.1. *Let $u \in W^{1,p}(\Omega)$ and $f \in L^{p,\lambda}(\Omega)$, with $2 \leq p < n$, $n - p < \lambda < n$. If $\nabla u \in L^{p,\eta}(\Omega)$ for some $\eta \in [0, n - p]$, then*

$$fu \in L^{p,\lambda+\eta-n+p}(\Omega).$$

Moreover there exists a positive constant C , independent of u and f , such that

$$\|fu\|_{L^{p,\lambda+\eta-n+p}(\Omega)} \leq C\|f\|_{L^{p,\lambda}(\Omega)}(\|\nabla u\|_{L^{p,\eta}(\Omega)} + \|u\|_{L^p(\Omega)}).$$

The lemma has been proved in [5] (Lemma 4.1) for $p = 2$. The extension to the case $p \neq 2$ is straightforward.

3. HÖLDER REGULARITY

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying the condition K .

Let us consider the following linear second order elliptic equation in non divergence form

$$(2) \quad a_{ij}u_{x_i x_j} + b_i u_{x_i} + cu = f,$$

where we assume

$$(3) \quad a_{ij}(x) = a_{ji}(x) \quad a.e. \ x \in \Omega, \ i, j = 1, \dots, n,$$

$$(4) \quad \exists \mu > 0 : \mu|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i \xi_j \leq \frac{1}{\mu}|\xi|^2, \quad a.e. \ x \in \mathbb{R}^n, \ \forall \xi \in \Omega,$$

and

$$(5) \quad a_{ij} \in L^\infty(\Omega) \cap VMO(\Omega), \ c, \ b_i, \ f \in L^p(\Omega).$$

Definition 3.1. *A function u in $W^{2,p}(\Omega)$ is a strong solution of the equation (2) if u satisfies (2) a.e. $x \in \Omega$.*

Here we recall the Theorem 3.3 in [7] regarding the equation (2) with $b_i = 0$ and $c = 0$.

Theorem 3.1. *Let (3), (4), (5) hold true. Let $u \in W^{2,p} \cap W_0^{1,p}(\Omega)$ be a strong solution of (6), with $b_i = 0$ and $c = 0$ and f belong to $L^{p,\lambda}(\Omega)$, $p > 1$, $0 < \lambda < n$. Then $D^2u \in L^{p,\lambda}(\Omega)$ and there exists a positive constant C such that*

$$\|D^2u\|_{L^{p,\lambda}(\Omega)} \leq C \left(\|u\|_{L^{p,\lambda}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)} \right).$$

Our first result concerns the case $b_i = 0$, $i = 1, 2, \dots, n$, i.e. we consider the equation

$$(6) \quad a_{ij}u_{x_i x_j} + cu = f.$$

Regarding the equation (6) we prove the following result.

Theorem 3.2. *Let (3), (4), (5) hold true and let $u \in W^{2,p} \cap W_0^{1,p}(\Omega)$ be a strong solution of (6), c , and f belong to $L^{p,\lambda}(\Omega)$, $1 < p < n$, $n - p < \lambda < n$. Then, for all $0 < \epsilon < \min\{n - p, p + \lambda - n\}$ we have that $\nabla u \in C^{0,\gamma}(\overline{\Omega})$, where $\gamma = 1 - \frac{n-\lambda+\epsilon}{p}$. Moreover there exists a positive constant C such that*

$$[\nabla u]_\gamma \leq C \left(\|c\|_{L^{p,\lambda}(\Omega)} \|\nabla u\|_{L^{p,n-p-\epsilon}(\Omega)} + (\|c\|_{L^{p,\lambda}(\Omega)} + 1) \|u\|_{L^{p,\lambda-\epsilon}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)} \right).$$

Proof. We start by noting that $\nabla u \in W^{1,p}(\Omega)$ implies $\nabla u \in L^{p,p}(\Omega)$.

If $p \geq n - p$ then $\nabla u \in L^{p,n-p-\epsilon}(\Omega)$, with $0 < \epsilon < \min\{n - p, p + \lambda - n\}$, and we apply Lemma 2.1 to obtain $cu \in L^{p,\lambda-\epsilon}(\Omega)$, from which and Theorem 3.1 we obtain $D^2u \in L^{p,\lambda-\epsilon}(\Omega)$. Then, from Poincaré inequality $\nabla u \in \mathcal{L}^{p,p+\lambda-\epsilon}(\Omega)$ and since $p + \lambda - \epsilon > n$ we obtain that $\nabla u \in C^{0,\gamma}(\overline{\Omega})$.

If $p < n - p$ from Lemma 2.1 we obtain $cu \in L^{p,\lambda+2p-n}(\Omega)$, from which and Theorem 3.1, since $\lambda + 2p - n < \lambda$ we obtain that $D^2u \in L^{p,\lambda+2p-n}(\Omega)$ and consequently $\nabla u \in \mathcal{L}^{p,\lambda+3p-n}(\Omega)$.

If $\lambda + 3p - n > n$ then $\nabla u \in C^{0,\gamma}(\overline{\Omega})$.

If $\lambda + 3p - n < n$ $\nabla u \in L^{p,\lambda+3p-n}(\Omega)$. If also $\lambda + 3p - n \geq n - p$ then $\nabla u \in L^{p,n-p-\epsilon}(\Omega)$, with $0 < \epsilon < \min\{n - p, p + \lambda - n\}$, and we apply Lemma 2.1 to obtain $cu \in L^{p,\lambda-\epsilon}(\Omega)$, from which and Theorem 3.1 we obtain $D^2u \in L^{p,\lambda-\epsilon}(\Omega)$. Then $\nabla u \in \mathcal{L}^{p,p+\lambda-\epsilon}(\Omega)$ and since $p + \lambda - \epsilon > n$ we obtain that $\nabla u \in C^{0,\gamma}(\overline{\Omega})$.

If $\lambda + 3p - n < n - p$ then from Lemma 2.1 we obtain $cu \in L^{p,2\lambda+4p-2n}(\Omega)$, from which and Theorem 3.1, since $2\lambda + 4p - n < \lambda$, we obtain that $D^2u \in L^{p,2\lambda+4p-2n}(\Omega)$ and $\nabla u \in \mathcal{L}^{p,2\lambda+5p-2n}(\Omega)$.

If $2\lambda + 5p - 2n > n$ then $\nabla u \in C^{0,\gamma}(\overline{\Omega})$.

If $2\lambda + 5p - 2n < n$ $\nabla u \in L^{p,2\lambda+5p-2n}(\Omega)$ we can proceed as in the case $\lambda + 3p - n < n$ and so on.

Finally there exists $k \in \mathbb{N}$ such that $n - p \leq (2k + 1)p + k(\lambda - n)$, and $\nabla u \in L^{p,(2k+1)p+k(\lambda-n)}(\Omega)$ then $\nabla u \in C^{0,\gamma}(\overline{\Omega})$. Moreover from Theorem 2.1,

Theorem 3.1 and Lemma 2.1 we get

$$\begin{aligned}
[\nabla u]_\gamma &\leq C[\nabla u]_{\mathcal{L}^{p,p+\lambda-\epsilon}(\Omega)} \leq C\|D^2u\|_{L^{p,\lambda-\epsilon}(\Omega)} \leq \\
&\leq C\{\|cu\|_{L^{p,\lambda-\epsilon}(\Omega)} + \|u\|_{L^{p,\lambda-\epsilon}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\} \leq \\
&\leq C\{\|c\|_{L^{p,\lambda}(\Omega)}[\|\nabla u\|_{L^{p,n-p-\epsilon}(\Omega)} + \\
&\quad + \|u\|_{L^p(\Omega)}] + \|u\|_{L^{p,\lambda-\epsilon}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\} \leq \\
&\leq C\{\|c\|_{L^{p,\lambda}(\Omega)}\|\nabla u\|_{L^{p,n-p-\epsilon}(\Omega)} + \\
&\quad + (\|c\|_{L^{p,\lambda}(\Omega)} + 1)\|u\|_{L^{p,\lambda-\epsilon}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\}.
\end{aligned}$$

□

Now we study

$$(7) \quad a_{ij}u_{x_i x_j} + b_i u_{x_i} = f.$$

Theorem 3.3. *Let (3), (4), (5) hold true. Let $u \in W^{2,p} \cap W_0^{1,p}(\Omega)$ be a strong solution of (7), b_i , $i = 1, \dots, n$ and f belong to $L^{p,\lambda}(\Omega)$, $p < n$, $n - p < \lambda < n$. Then there exists $k \in \mathbb{N}$ such that $k\lambda - kn + (k+1)p > n$ and $\nabla u \in C^{0,\gamma}(\overline{\Omega})$ with $\gamma = \frac{k\lambda - (k+1)n + (k+1)p}{p}$. Moreover there exists a positive constant C depending on $\|b\|_{L^{p,\lambda}(\Omega)}$ and k such that*

$$(8) \quad [\nabla u]_\gamma \leq C \left(\|D^2u\|_{L^p(\Omega)} + \|u\|_{L^{p,\lambda-n+p}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)} \right).$$

Proof. Since $D^2u \in L^p(\Omega)$ from Lemma 2.1 we obtain that $b \cdot \nabla u \in L^{p,\lambda-n+p}(\Omega)$. Then from Theorem 3.1 $D^2u \in L^{p,\lambda-n+p}(\Omega)$, and we have

$$\begin{aligned}
(9) \quad \|D^2u\|_{L^{p,\lambda-n+p}(\Omega)} &\leq \\
&\leq C\{\|b \nabla u\|_{L^{p,\lambda-n+p}(\Omega)} + \|u\|_{L^{p,\lambda-n+p}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\} \leq \\
&\leq C\{\|b\|_{L^{p,\lambda}(\Omega)}[\|D^2u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}] + \|u\|_{L^{p,\lambda-n+p}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\} \leq \\
&\leq C\{\|b\|_{L^{p,\lambda}(\Omega)}\|D^2u\|_{L^p(\Omega)} + \\
&\quad + (\|b\|_{L^{p,\lambda}(\Omega)} + 1)\|u\|_{L^{p,\lambda-n+p}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\}.
\end{aligned}$$

From Poincaré inequality $\nabla u \in \mathcal{L}^{p,\lambda-n+2p}(\Omega)$. If $\lambda - n + 2p > n$ we have that $\nabla u \in C^{0,\gamma}(\overline{\Omega})$ with $\gamma = \frac{\lambda - 2n + 2p}{p}$. So, from Theorem 2.1, Poincaré inequality and

(9)

$$\begin{aligned}
[\nabla u]_\gamma &\leq [\nabla u]_{\mathcal{L}^{p,\lambda-n+2p}(\Omega)} \leq \|D^2u\|_{L^{p,\lambda-n+p}(\Omega)} \leq \\
&\leq C\{\|b\|_{L^{p,\lambda}(\Omega)}\|D^2u\|_{L^p(\Omega)} + \\
&\quad + (\|b\|_{L^{p,\lambda}(\Omega)} + 1)\|u\|_{L^{p,\lambda-n+p}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\}.
\end{aligned}$$

If $\lambda - n + 2p < n$, since $\lambda - n + p < n - p$, we can apply Lemma 2.1 to obtain $b \nabla u \in L^{p,2\lambda-2n+2p}(\Omega)$. Since $2\lambda - 2n + 2p < \lambda$ from Theorem 3.1

$D^2u \in L^{p,2\lambda-2n+2p}(\Omega)$ and we get from also (9)

$$\begin{aligned}
\|D^2u\|_{L^{p,2\lambda-2n+2p}(\Omega)} &\leq \\
&\leq C\{\|b\nabla u\|_{L^{p,2\lambda-2n+2p}(\Omega)} + \|u\|_{L^{p,2\lambda-2n+2p}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\} \leq \\
&\leq C\{\|b\|_{L^{p,\lambda}(\Omega)}[\|D^2u\|_{L^{p,\lambda-n+p}(\Omega)} + \|u\|_{L^p(\Omega)}] + \\
&\quad + \|u\|_{L^{p,2\lambda-2n+2p}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\} \leq \\
&\leq C\{\|b\|_{L^{p,\lambda}(\Omega)}^2\|D^2u\|_{L^p(\Omega)} + (\|b\|_{L^{p,\lambda}(\Omega)}^2 + \\
&\quad + \|b\|_{L^{p,\lambda}(\Omega)} + 1)\|u\|_{L^{p,\lambda-n+p}(\Omega)} + (\|b\|_{L^{p,\lambda}(\Omega)} + 1)\|f\|_{L^{p,\lambda}(\Omega)}\}.
\end{aligned}$$

Now from Poincaré inequality we obtain $\nabla u \in \mathcal{L}^{p,2\lambda-2n+3p}(B_2)$.

If $2\lambda - 2n + 3p > n$ $\nabla u \in C^{0,\gamma}(\overline{\Omega})$, with $\gamma = \frac{2\lambda-3n+3p}{p}$, and

$$\begin{aligned}
[\nabla u]_\gamma &\leq [\nabla u]_{\mathcal{L}^{p,2\lambda-2n+3p}(\Omega)} \leq \|D^2u\|_{L^{p,2\lambda-2n+2p}(\Omega)} \leq \\
&\leq C\{\|b\|_{L^{p,\lambda}(\Omega)}^2\|D^2u\|_{L^p(\Omega)} + \\
&\quad + (\|b\|_{L^{p,\lambda}(\Omega)}^2 + \|b\|_{L^{p,\lambda}(\Omega)} + 1)\|u\|_{L^{p,\lambda-n+p}(\Omega)} + \\
&\quad + (\|b\|_{L^{p,\lambda}(\Omega)} + 1)\|f\|_{L^{p,\lambda}(\Omega)}\}.
\end{aligned}$$

If $2\lambda - 2n + 3p < n$ ($2\lambda - 2n + 2p < n - p$) we proceed as the previous cases.

Finally there exists a positive integer k such that $k\lambda - kn + (k+1)p > n$ then $\nabla u \in C^{0,\gamma}(\overline{\Omega})$ with $\gamma = \frac{k\lambda-(k+1)n+(k+1)p}{p}$ and

$$\begin{aligned}
[\nabla u]_\gamma &\leq [\nabla u]_{\mathcal{L}^{p,k\lambda-kn+(k+1)p}(\Omega)} \leq \|D^2u\|_{L^{p,k\lambda-kn+(k+1)p}(\Omega)} \leq \\
&\leq C\{\|b\|_{L^{p,\lambda}(\Omega)}^k\|D^2u\|_{L^p(\Omega)} + \\
&\quad + (\|b\|_{L^{p,\lambda}(\Omega)}^k + \|b\|_{L^{p,\lambda}(\Omega)}^{k-1} + \cdots + \|b\|_{L^{p,\lambda}(\Omega)} + 1)\|u\|_{L^{p,\lambda-n+p}(\Omega)} + \\
&\quad + (\|b\|_{L^{p,\lambda}(\Omega)}^{k-1} + \cdots + \|b\|_{L^{p,\lambda}(\Omega)} + 1)\|f\|_{L^{p,\lambda}(\Omega)}\}
\end{aligned}$$

from which (8) follows. \square

The techniques used in Theorems 3.2 and 3.3 allow also to prove the Hölder regularity for the gradient of the solutions $u \in W^{2,p} \cap W_0^{1,p}(\Omega)$ of the complete equation (2).

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